Homogeneous Solutions of Fully Nonlinear Elliptic Equations in Four Dimensions

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Abstract. We prove that there is no nontrivial homogeneous order 2 solutions of fully nonlinear uniformly elliptic equations in dimension 4.

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1 Introduction

We study a class of solutions to fully nonlinear second-order elliptic equations of the form

$$F(D^2u) = 0 \tag{1}$$

D^2u being the Hessian of the function u defined in $\mathbb{R}^n$. We assume that F is a smooth function defined on the space $S^2(\mathbb{R}^n)$ of $n \times n$ symmetric matrices satisfying the uniform ellipticity condition:

$$\frac{1}{C'}|\xi|^2 \le F_{u_{ij}}\xi_i\xi_j \le C'|\xi|^2 \ , \forall \xi \in \mathbf{R}^n \ .$$

Here, u_{ij} denotes the partial derivative $\partial^2 u/\partial x_i \partial x_j$. A function u is called a *classical* solution of (1) if $u \in C^2(\Omega)$ and u satisfies (1). Actually, any classical solution of (1) is a smooth ($C^{\alpha+3}$) solution, provided that F is a smooth (C^α) function of its arguments.

Let $B = \{x \in \mathbb{R}^n : |x| < 1\}$ be a ball, g be a continuous function on ∂B. Consider a Dirichlet problem

$$\begin{cases} F(D^2u) = 0 & \text{in } B \\ u = g & \text{on } \partial B \end{cases} \tag{2}$$

We are interested in the problem of existence and regularity of solutions to the Dirichlet problem (2). The problem (2) has always a unique viscosity (weak) solution for fully nonlinear elliptic equations. The viscosity solutions satisfy the equation (1) in a weak sense, and the best known interior regularity ([C],[CC],[T3]) for them is $C^{1,\varepsilon}$ for some $\varepsilon > 0$. For more details see [CC], [CIL].

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Note, however, that viscosity solutions are $C^{2,\varepsilon}$-regular almost everywhere; in fact, it is true on the complement of a closed set of Hausdorff dimension strictly less then n [ASS]. Until recently it remained unclear whether non-smooth viscosity solutions exist. In the recent papers [NV1], [NV2], [NV3], [NV4] the authors first proved the existence of non-classical viscosity solutions to a fully nonlinear elliptic equation, and of singular solutions to Hessian (i.e. dependinding only on the eigenvalues of D^2u) uniformly elliptic equation in all dimensions beginning from 12, and, finally, the paper [NTV] gives a construction of non-smooth viscosity solution in 5 dimensions which is order 2 homogeneous, also for Hessian equations. These papers use the functions

$$w_5(x) = \frac{P_5(x)}{|x|},\ w_{12,\delta}(x) = \frac{P_{12}(x)}{|x|^\delta},\ w_{24,\delta}(x) = \frac{P_{24}(x)}{|x|^\delta},\ \delta \in [1, 2[,$$

for certain (minimal) cubic forms $P_5(x), P_{12}(x), P_{24}(x)$ in the dimensions 5,12 and 24, respectively.

On the other hand the classical Alexandrov's theorem [A] says that an analytic in $\mathbb{R}^3 \setminus \{0\}$ homogeneous order 1 function u such that the Hessian D^2u is either non-definite or 0 at any point is linear. This immediately implies the absense of homogeneous order 2 real analytic in $\mathbb{R}^3 \setminus \{0\}$ solutions to fully nonlinear equations different from quadratic forms (in $C^{2,\alpha}$ setting it is proved in [HNY]). Thus the existence of homogeneous order 2 real analytic outside zero solutions to fully nonlinear equations is not known exactly in 4 dimensions, the analogue of Alexandrov's theorem in 4 dimensions being false (indeed $u = (x_1^2 + x_2^2 - x_3^2 - x_4^2)/|x|$ gives a counter-example, cf. [LO]).

This note fills this gap showing that 5 is the minimal dimension where there exist homogeneous order 2 non-smooth solutions to uniformly elliptic fully nonlinear equations.

Theorem 1. *Let u be a homogeneous order 2 real analytic function in $\mathbb{R}^4 \setminus \{0\}$. If u is a solution of the uniformly elliptic equation $F(D^2u) = 0$ in $\mathbb{R}^4 \setminus \{0\}$, then u is a quadratic polynomial.*

We collect some preliminary lemmas in Section 2 below and give the proof in Section 3.

2 Preliminary results

Here we prove some general results we need to prove the theorem.

Lemma 0. *Let v be a smooth homogeneous order 1 function in $\mathbb{R}^3 \setminus \{0\}$. Assume that $y \in \mathbb{S}^2$ and the quadratic form $D^2v(y)$ changes sign. Let $a \in \mathbb{S}^3, a \neq y$, and let G be an open domain in $\mathbb{R}^3, y \in G$. Then*

$$\sup_G v_a(x) > v_a(y).$$

Proof. Let $L \subset \mathbb{R}^3$ be an affine 2-dimensional plane transversal to the vector y such that $y \in L$ and a is parallel to L. Denote by v' the restriction of the function v on L. Since v is a homogeneous order 1 function the quadratic form $D^2v'(y)$ changes sign. Thus there is a neighborhood D of the point y where v' satisfies a uniformly elliptic equation on L of the form

$$\sum a_{ij}(x)\frac{\partial^2 v'}{\partial x_i \partial x_j} = 0.$$

Thus by the maximum principle for the gradient of a solution of elliptic equations in dimension 2, see [GT], v'_a cannot attain the supremum at the point y. The lemma is proved.

Lemma 1. *Let v be a real analytic homogeneous order 1 function in $\mathbb{R}^n \setminus \{0\}$. Assume that v is a solution of a linear uniformly elliptic equation*

$$Pv = \sum a_{ij}(x/|x|)\frac{\partial^2 v}{\partial x_i \partial x_j} = 0,$$

where coefficients a_{ij} are smooth functions on $\mathbb{S}^{n-1}$. Let $e_1, ..., e_n \in \mathbb{S}^{n-1}$ be linearly independent unit vectors. Assume that the functions v_{e_i}, $i = 1, ..., n$ attain local supremum at $a \in \mathbb{S}^{n-1}, a \neq e_i, i = 1, ..., n$. Then v is a linear function.

Proof. Denote by L an affine hyperplane in $\mathbb{R}^n$ orthogonal to a, $a \in L$. Then the restriction v' of the function v on L satisfies a linear uniformly elliptic equation of the type

$$P(v') = \sum a'_{ij}(y)\frac{\partial^2 v'}{\partial x_i \partial x_j} = 0,$$

where $y \in L$ and a'_{ij} are smooth functions on L. Indeed, $D^2v(a) = 0$ since v is order one homogeneous, thus the partial derivatives of v' coinside with ones of v in an appropriate coordinate system. We consider then a coordinate system on L such that the point a becomes the origin, assuming without loss that $v'(0) = 0, \nabla v'(0) = 0$. After a linear transformation of $\mathbb{R}^n$ we can assume that $P(0)$ is the Laplacian, i.e., $a'_{ij}(0) = \delta_i^j$. Let p, $\deg p = k \geq 2$ be the first nonzero homogeneous polynomial of the Taylor expansion of v' at 0; clearly p is harmonic. Let $B \subset L$ be a small ball centered at 0, let g be the gradient map

$$g : L \to \mathbb{R}^{n-1},\ g := \nabla v'$$

and let $\Gamma = g(B)$. Then $\Gamma \subset K := \bigcap_{i=1}^n \{e_i \leq 0\}$, K being a strictly convex cone in $\mathbb{R}^n$ since e_i are linearly independent. Denote $K_0 = \{K + a\} \cap L$; if K_0 is non-empty then K_0 is a strictly convex cone in L. Let p' be a non-zero partial derivative of p of order $k-2$; the quadratic form p' changes sign, hence $\nabla p'(L)$ intersects the complement of K_0 and thus $l^+ \bigcap K_0 = \emptyset$ for a line $l \subset \nabla p'(L)$ and a ray $l^+ \subset l$. Let $\Lambda := \nabla {p'}^{-1}(l^+)$, then the curve $g(\Lambda) \subset \mathbb{R}^n$ is tangent to

l^+ at the point $\{a\}$ since $v_a(x) = O(|a-x|^k)$. Therefore $g(\Lambda \cap B)$ intersects the complement of K, and the lemma follows.

Lemma 2. *Let v be a real analytic homogeneous order 1 function in $\mathbb{R}^4 \setminus \{0\}$. Assume that v is a solution of a linear uniformly elliptic equation*

$$Pv = \sum a_{ij}(x/|x|) \frac{\partial^2 v}{\partial x_i \partial x_j} = 0, \tag{3}$$

and the rank of the gradient map $\nabla v : \mathbb{S}^3 \to \mathbb{R}^4$ is ≤ 2. Then v is a linear function.

Proof. Let $y \in \mathbb{S}^3$, $m \subset \mathbb{R}^4$ be a subspace, $m \perp y$. Let $M \subset R^4$ be an affine hyperplane parallel to m, $y \in M$, and let f be the restriction of v on M. Then f is a real analytic function on M such that for any $x \in M$ the hessian $D^2 f(x)$ is degenerate and either the quadratic form $D^2 f(x)$ changes sign or $D^2 f(x) = 0$. Let

$$H := \{x \in \mathbb{R}^3 : \text{rank}(D^2 f(x)) = 2\}.$$

We assume without loss that $codim(\mathbb{R}^3 \setminus H) \geq 1$. For $x \in H$ let $z(x)$ be the zero eigenspace of $D^2 f(x)$. By assumption of the lemma $z(x)$ is a line analytically depending on the point $x \in H$. By Chern-Lashof's lemma, [CL, Lemma 2], [S, Lemma VI 5.1] in the neighborhood of any point $x \in M$ the plane M is foliated by a 2-dimensional family of straight lines L, such that for any line $l \in L$ the restriction of the function f on l is an affine function, moreover l is parallel to the line $z(x)$ at any point $x \in l$, see the proof of Lemma 2 in [CL]. By the analyticity of f it follows that the family L foliate the whole space M without intersection. Let $l \in L$ and $p \subset \mathbb{R}^4$ be a two-dimensional plane spanned by l in $\mathbb{R}^4$. Since v is a homogeneous order one function it follows that v is linear on a half-plane of p. By analyticity, v is a linear function on the whole plane p. Denote the whole set of these planes p by P. Then any two planes of P intersect only at $\{0\}$ and foliate $\mathbb{R}^4 \setminus m$.

Let $y' \in \mathbb{S}^3$, $m' = (y')^\perp \subset \mathbb{R}^4$ and let P' be the foliation of $\mathbb{R}^4 \setminus m'$ by two-dimensional planes corresponding to y'. We will prove that P and P' coincide on $\mathbb{R}^4 \setminus (m \cup m')$. Assume not. Then there is a 4-dimensional subset $X \subset \mathbb{R}^4$ such that for any $x \in X$ one has $x \in p \cap p'$ for some $p \in P$, $p' \in P'$, $p \neq p'$. Since the planes p and p' are zero eigenspaces of $D^2 v$ it follows that that the zero eigenvalue has multiplicity at least 3 at x, and hence $D^2 v(x) = 0$. Thus $D^2 v$ vanishes on X and hence by analyticity of v it follows that v is a linear function. Thus choosing different $y \in \mathbb{S}^3$ we get a foliation P of $\mathbb{R}^4 \setminus \{0\}$ by two dimensional planes which are zero eigenspaces of $D^2 v$.

Notice that any 3-dimensional subspace of $\mathbb{R}^4$ contains at most one plane of P, since any two different planes in 3-dimensional space have nontrivial intersection.

Let $m \in \mathbb{R}^4$ be a 3-dimensional subspace such that $m \supset p$, $p \in P$. Denote by v' the restriction of the function of v to m; subtracting a linear function we can assume that $v' = 0$ on p. Let $x \in m \setminus p$, $x \in p'$ for some $p' \in P$. Then p' is trasversal to m. Since p' is a zero eigenspace of $D^2 v$ it follows that either

$D^2v'(x)$ changes sign or $D^2v'(x) = 0$ on m. Thus the function v' is a solution of an elliptic equation (3) at x. Thus we proved that v' satisfies an elliptic equation (3) on $m \setminus p$. Let $e \in m$ be a vector parallel to p. Let $z \in \mathbb{S}^2 \subset m$ be a point at which v'_e attains its maximum on $\mathbb{S}^2$. If $v'_e(z) > 0$, then $z \in \mathbb{S}^2 \setminus p$ since by our assumption $v' = 0$ on p. Since in a neighborhood of z the function v' is a solution of (3) this contradicts Lemma 0. Thus $v'_e(z) \leq 0$ and thus $v'_e \leq 0$ everywhere since $v'_e(z)$ is maximal. Applying the same argument to the function $-v'$ we get $v'_e \geq 0$ everywhere and thus $v'_e \equiv 0$ for any vector e parallel to p. Hence v' is a function which depends only on the coordinate orthogonal to p and therefore v' is a linear function. Thus we get that for any three dimensional subspace m of $\mathbb{R}^4$ the restriction of v on m is a linear function. Hence v is a linear function on $\mathbb{R}^4$ and the lemma is proved.

Lemma 3. *Let $Q(x, y, z) \in \mathbb{R}[x, y, z]$ be a cubic form such that for any $e \in \mathbb{S}^2$ the quadratic form Q_e is degenerate. Then Q is a function of two variables in some coordinate system.*

Proof. First of all, the conditions as well as the conclusion of the lemma are invariant under non-singular linear transformations. Considering $Q(x, y, z) = 0$ as an equation of a plane projective cubic curve E_Q and applying the usual argument giving its Weierstrrass form (see, e.g. pp. 45-46 in the proof of Proposition 1.2 of Ch. 2 in [M]) one gets one the following:

1. E_Q is elliptic or irreducible possesing a singular point with $y \neq 0$; in this case Q is equivalent under a linear transfomation to the Weierstrass form

$$Q_W = y^2z + x^3 + px^2z + qz^3;$$

2. E_Q is irreducible possesing a singular point with $y = 0$; then

$$Q = Q_s = x^3 + axyz + bxz^2 + cyz^2 + dz^3$$

after a suitable non-singular linear transformation;

3. E_Q is reducible, then either

$$Q = Q_r = z(x^2 + ay^2 + bz^2 + cxz + dyz)$$

modulo such a transformation or Q verifies the conclusion.

If $Q = y^2z + x^3 + px^2z + qz^3, e = (k, l, m)$ then

$$r = r(k, l, m) := \det(D^2(Q_e)) = 3k^2mp + 9km^2q - 3kl^2 - m^3p^2$$

should be indentically zero; in particular, $-6 = r_{kll} = \partial^3 r/\partial k \partial l^2 = 0$ which is clearly not the case.

If $Q = x^3 + axyz + bxz^2 + cyz^2 + dz^3$ then

$$r/2 = a^3klm + a^2bkm^2 + a^2clm^2 - 3a^2dm^3 + 4abcm^3 - 3a^2k^3 - 12ack^2m - 12c^2km^2,$$

$$0 = r_{klm}/2 = a^3,\ 0 = r_{kmm}/4 = a^2b - 12c^2$$

implying $c = a = 0$ and the conclusion.

If $Q = z(x^2 + ay^2 + bz^2 + cxz + dyz)$ then

$$r/8 = 3abm^3 - ac^2m^3 - a^2l^2m - ackm^2 - adlm^2 - d^2m^3 - ak^2m,$$

$$0 = r_{llm}/16 = -a^2,\ 0 = r_{mmm}/48 = 3ab - ac^2 - d^2$$

thus $a = d = 0$ as necessary and the proof is finished.

Lemma 4. *Let $Q(x,y,z) \in \mathbb{R}[x,y,z]$ be a cubic form such that for any $a \neq b \in C \subset \mathbb{S}^2$ the partial derivative Q_{ab} vanishes as a linear form, C being a curve on $\mathbb{S}^2$. Then Q is a function of two variables in some coordinate system.*

Proof. The proof is very similar to that of Lemma , but sightly more combersome. We consider the same three main cases, each of them being divided in subcases depending on the curve $C \subset \mathbb{S}^2$.

1). Weierstrass case. There are two subcases:

1a). The curve C is not in $\mathbb{S}^2 \bigcap (\{y=0\} \bigcup \{z=0\})$.

1b). The curve $C \subset \mathbb{S}^2 \bigcap (\{y=0\} \bigcup \{z=0\})$.

In the subcase 1a we can suppose without loss that $a = (a_1, b_1, c_1)$, $b = (a_2, b_2, c_2)$ with $c_1b_2 + c_2b_1 \neq 0$. A brute force calculation gives $Q_{aby}/2 = c_1b_2 + c_2b_1 \neq 0$ and thus we get a contradiction.

In the subcase 1b we suppose without loss that $a = (a_1, b_1, 0)$, $b = (a_2, b_2, 0)$ with $a_1a_2 \neq 0$ but then $Q_{abx}/6 = a_1a_2 \neq 0$.

2). Singular case (singularity at $y = 0$), $Q = x^3 + pxyz + qxz^2 + ryz^2 + sz^3$. Subcases:

2a). The curve C is not in $\mathbb{S}^2 \bigcap \{z=0\}$.

2b). The curve $C \subset \mathbb{S}^2 \bigcap \{z=0\}$.

Suppose 2a, $a = (a_1, b_1, c_1)$, $b = (a_2, b_2, c_2)$, $c_1c_2 \neq 0$. Then the condition $Q_{abx} = 0$ implies $2c_1c_2r = -(a_2c_1 + c_2a_1)p$. If there exists $c = (a_3, b_3, c_3) \in C$ such that $c_3a_2 \neq a_3c_2$ then $0 = Q_{acy} = -c_1p(c_3a_2 - a_3c_2)/c_2$ gives $p = 0, r = 0$ which proves the lemma. If $a_3c_2 = a_2c_3$ we can suppose that $b_3c_2 \neq c_3b_2,$ and the condition $0 = Q_{acx} = c_1p(c_2b_3 - b_2c_3)/c_2$ gives $r = p = 0$.

In the case 2b we get $a = (a_1, b_1, 0), b = (a_2, b_2, 0), a_1a_2 \neq 0$, and hence $Q_{abx} = 3a_1a_2 \neq 0$.

3). Reducible case, $Q = z(x^2 + py^2 + qz^2 + rxz + syz)$. Subcases:

3a). The curve C is not in $\mathbb{S}^2 \bigcap \{z=0\}$.

3b). The curve $C \subset \mathbb{S}^2 \bigcap \{z=0\}$.

Suppose 3a, $a = (a_1, b_1, c_1)$, $b = (a_2, b_2, c_2)$, $c_1c_2 \neq 0$. Then the condition $Q_{aby} = 0$ implies that $c_1c_2s = -(b_2c_1 + c_2b_1)p$. For any $c = (a_3, b_3, c_3)$ one gets $0 = Q_{acy} = p(b_3c_2 - c_3b_2)c_1/c_2$ with $b_3c_2 \neq c_3b_2$ since $b_3c_2 = c_3b_2$ gives $Q_{acx} = (a_3c_2 - c_3a_2)c_1/c_2 \neq 0$. Hence $s = p = 0$.

Suppose 3b, $a = (a_1, b_1, 0)$, $b = (a_2, b_2, 0)$, $c = (a_3, b_3, 0)$, $a_1a_2 \neq 0$, $b_1b_2 \neq 0$, $a_2b_3 \neq a_3b_2$. Then $0 = Q_{abz} = pb_1b_2 + a_1a_2$, $p = -a_1a_2/(b_1b_2)$, $Q_{acz} = a_1(a_3b_2 - b_3a_2)/b_2 \neq 0$, a contadiction and the proof is finished.

3 Proof of the Theorem

We begin with the following construction.

Let $x \in \mathbb{S}^3$. Set

$$A_x = \{(a,b) \in \mathbb{S}^3 \times \mathbb{S}^3, a \neq b : u_{a,b}(x) = \sup_{y \in \mathbb{S}^3} u_{a,b}(y)\};$$

note that A_x is a semi-analytic subset of $\mathbb{S}^3 \times \mathbb{S}^3$, and $(a,b) \in A_x$ implies $(b,a) \in A_x$. The semi-analycity of A_x implies the sub-analycity of all the sets below in the proof. In particular they verify Whitney's stratification theorem [W] as was showed by Hironaka [H], i.e. each such set M is stratified in a finite union of open k-dimensional smooth submanifolds, $k = 0, 1, ..., m = \dim M$.

Let then $\mathfrak{C}^x$ for $x \in \mathbb{R}^4 \setminus \{0\}$ be the cubic form of the Taylor expansion of the function u at the point x, i.e., $D^3\mathfrak{C}^x = D^3u(x)$. Let us notice first that for any vector $e \in \mathbb{R}^4$ the function u_e is a homogeneous order 1 and hence x is a zero eigenvector of the quadratic form $(\mathfrak{C}^x_e)$. We need the following two simple properties of this form.

Lemma 5. *Let $(a,b) \in A_x$. Then b is a zero eigenvector of the quadratic form $\mathfrak{C}^x_a$.*

Proof. From our assumptions it follows that for any vector $e \in \mathbb{R}^4$ one has $u_{a,b,e}(x) = 0$. Hence $(\mathfrak{C}^x_a)_{b,e} = 0$. This implies that b is a zero eigenvector of $\mathfrak{C}^x_a$.

Lemma 6. *Let $a, x, b_1, b_2, b_3 \in \mathbb{S}^3$ with linearly independent b_1, b_2, b_3 such that $(a,b_1), (a,b_2), (a,b_3) \in A_x$. Then $\mathfrak{C}^x_a = 0$.*

Proof. By Lemma 5 the vectors b_i are zero eigenvectors of the quadratic form $\mathfrak{C}^x_a$, i.e., it has the zero eigenvalue with multiplicity at least 3. Since $\mathfrak{C}^x_a$ should change the sign or be equal zero the lemma follows.

Let now

$$X := \{x \in \mathbb{S}^3 : \dim A_x \geq 3\}.$$

Then $X \neq \emptyset$ since

$$\bigcup_{x \in \mathbb{S}^3} A_x = \mathbb{S}^3 \times \mathbb{S}^3,\ \dim(\mathbb{S}^3 \times \mathbb{S}^3) = 6,$$

we denote $d \in [0,3]$ its dimension.

Let $\Gamma = \cup_{x \in X} A_x$ then $\dim(\mathbb{S}^3 \times \mathbb{S}^3 \setminus \Gamma) \leq 5, \dim(\Gamma) = 6$.

We have four possibilities for d, namely, $d = 0, 1, 2$ or 3.

1. Let $d = 0$. Then $\dim A_y = 6$ for some $y \in X$, and

$$\dim((\mathbb{S}^3 \times \{e\}) \bigcap A_y) \geq 3$$

for $e \in \mathbb{S}^3$.

In this case one can find linearly independent vectors $e_1, ..., e_4$, $e_i \neq y$, such that $(e, e_i) \in A_y$. Applying Lemma 1 to the function u_e we get the proof.

2. Let $d = 1$. Then we can suppose without loss that $\dim A_y = 5$ for any $y \in X$ and

$$\dim((\mathbb{S}^3 \times \{e\}) \bigcap A_y) \geq 2, \ \dim((\{e\} \times \mathbb{S}^3) \bigcap A_y) \geq 2$$

thus

$$E_1 \times E_2 \subset A_y$$

$E_1, E_2 \subset \mathbb{S}^3, \dim(E_1) = \dim(E_2) = 2$.

Denote the set of all $y \in \mathbb{S}^3$ satisfying $E_1 \times E_2 \subset A_y$ by Y. Let $y \in Y$, $a \in E_1$. Then By Lemma 6 $\mathfrak{C}^y_a = 0$. Since E_1 is a 2-dimensional set the cubic form $\mathfrak{C}^y$ depends at most on one coordinate. Since its derivative change sign it follows that $\mathfrak{C}^y = 0$. Thus if Y_1 is a connected component of Y then D^2u is constant on Y_1. On the other hand since Y is a real analytic set it contains only finite number of connected components, $Y_1, ..., Y_n$. At each Y_i function u has a fixed Hessian. Therefore there is at least one Y_j such that for $y \in Y_j$ the set A_y is 6-dimensional and one returns to the previous case.

3. Let $d = 2$. We suppose without loss that $\dim A_y = 4$ for any $y \in X$. For a connected component A, $\dim A = 4$ of A_y let $d_1 = d_1(A)$, $d_2 = d_2(A)$ be the dimensions of the projections of A to the first and the second factor in the product $\mathbb{S}^3 \times \mathbb{S}^3$ respectively. By symmetry one can suppose $d_1 \geq d_2 \geq 1$. Since $d_1 + d_2 \geq \dim A = 4$ we have the following possibilities:

3a). $d_1 = 3, d_2 = 1$;

3b). $d_1 = 2, d_2 = 2$;

3c). $d_1 = 3, d_2 = 2$;

3d). $d_1 = d_2 = 3$.

Since in the cases 3a and 3b one has $d_1 + d_2 = \dim A$, the manifold A itself is a product and we return to the cases 1 and 2 respectively.

Suppose 3c or 3d and let $Z \subset \mathbb{S}^3$ be the image of the first projection of A_x, $\dim Z = 3$. Then for any $x \in Z$ there is a curve $\gamma_x \subset \mathbb{S}^3$ verifying the following condition:

$$\forall a \in \gamma_x, \ a \times D(a) \subset A_x$$

for a 1- or 2-dimensional set $D(a) \subset \mathbb{S}^3$.

Let $y \in Z$, and let $a, a' \in \gamma_y, a \neq a'$. Then By Lemma 5 $\mathfrak{C}^y_a = 0$, $\mathfrak{C}^y_{a'} = 0$ and hence $\mathfrak{C}^y$ does not depend on the coordinates parallel to a and a'. Thus the cubic form $\mathfrak{C}^y$ depends at most on two coordinates. Thus for any $e \in \mathbb{S}^3$ the rank of the gradient map $\nabla \mathfrak{C}^y_e \to \mathbb{R}^4$ is at most 2 at the point $y \in Z$. Therefore since u_e is a homogeneous order one function the rank of the gradient map $\nabla_x u_e : \mathbb{S}^3 \to \mathbb{R}^4$ is at most 2 at any point $y \in Z$. For an affine hyperplane $L \subset \mathbb{R}^4, 0 \notin L$ let Z' be the spherical projection of Z on L, and let $s = u_e|_L$. Since u_e is a homogeneous order one function the gradient map of $u_e(x)$ depends only on the spherical coordinate of x it follows that $\det D^2 s = 0$ on Z'. Since s

is a real analytic function and Z' is a 3-dimensional we get $\det D^2 s = 0$ on the whole plane L and thus by Lemma 2 u_e is linear.

4. Let $d = 3$. We suppose without loss that $\dim A_y = 3$ for any $y \in X$. For a connected component $A, \dim A = 3$ of A_y let $d_1 \geq d_2$ be as before, $d_1 + d_2 \geq 3$. One has the following possibilities:

4a). $d_1 = 2, d_2 = 1$;

4b). $d_1 = d_2 = 2$;

4c). $d_1 = 3, d_2 = 0$;

4d). $d_1 = 3, d_2 = 1$;

4e). $d_1 = 3, d_2 = 2$;

4f). $d_1 = d_2 = 3$.

In the case 4a one has $A_x = E_1 \times C_2, \dim E_1 = 2, \dim C_2 = 1$ and the proof above for $A_x = E_1 \times E_2,\ \dim E_1 = \dim E_2 = 2$ remains valid.

In the case 4c one has $A_x = \mathbb{S}^3 \times \{a\}$ and we return to the case 1.

Suppose then 4d, 4e or 4f, let $Z_x := pr_1(A_x) \subset \mathbb{S}^3,\ \dim Z_x = 3$ Then for any $x \in X$ one gets:

$$\forall a \in Z_x,\ a \times h(a) \in A_x,$$

where $h(a) \in \mathbb{S}^3$.

Let $y \in X$ and let $L = y^\perp \subset \mathbb{R}^4$. Since u is a homogeneous order 2 function $\mathfrak{C}^y$ depends only on the coordinates of L. Thus there exists a 2-dimensional set $E \subset \mathbb{S}^2 \subset L$ such that $\mathfrak{C}^y_e$ is degenerate for any $e \in E$ and hence for any $e \in \mathbb{S}^2$. Thus by Lemma 3 the cubic form $\mathfrak{C}^y$ depends only on 2 variables and we finish the proof as for $d = 2$.

Assume finally 4b, and let $y \in X$.

Then by Lemma 4 the cubic form $\mathfrak{C}^y$ depends only on 2 coordinates, which we denote by z_1, z_2; let l be the linear span of z_1, z_2. Thus l is a zero eigenspace of $\mathfrak{C}^y_e$ for any $e \in \mathbb{S}^3$. By our assumption one finds $(a, b) \in A_y,\ b \notin l$. Therefore the multiplicity of the zero eigenvalue of $\mathfrak{C}^y_a$ is at least 3. Again, since its derivatives change sign it follows that $\mathfrak{C}^y = 0$ and one finishes the proof as before.

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